§7.1 Parallel Translation and Connections

Definition Let M be an oriented Riemannian manifold of dimension 2. Let $T(M)$ denote the tangent bundle of M. Let

$$S(M) = \{(m,\, v) \in T(M) \mid \langle v,\, v\rangle = 1\}$$

$S(M)$ is called the *sphere bundle*, or *circle bundle*, of M.

The notation $(m,\, v)$ for a point of $T(M)$ (or $S(M)$) is redundant since $v \in T(M,\, m)$. Nevertheless, we use it to emphasize that v is a tangent vector at m.

Remarks

(1) $S(M)$ is a smooth manifold ofdimension 3. The function $f : T(M) \to \mathbb{R}^1$, given by $f(m,\, v) = \langle v,\, v\rangle$, is smooth, and $df \neq 0$ whenever $f = 1$, so the implicit function theorem applies.

(2) Note that the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is a group under (complex) multiplication. Since $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$, the group S^1 is just the group of rotations of the oriented plane $\mathbb{R}^2$. This group *acts* on $S(M)$: there exists a smooth map

$$A : S^1 \times S(M) \to S(M)$$

given by

$$A(g,\, (m,\, v)) = (m.\, gv) \quad \text{for } g \in S^1,\ (m,\, v) \in S(M),$$

where gv is the image of the vector v under rotation by g in the oriented plane $T(M,\, m)$ (Figure 7.1). So, if $g = e^{i\theta}$, $\{v_1,\, v_2\}$ is any oriented orthonormal basis for $T(M,\, m)$, and if $v = c_1 v_1 + c_2 v_2$ for some $c_1,\, c_2 \in \mathbb{R}^1$, then, since $gv_1 = \cos\theta\, v_1 + \sin\theta\, v_2$ and $gv_2 = -\sin\theta\, v_1 + \cos\theta\, v_2$, we have

$$gv = (c_1\, \cos\theta - c_2\, \sin\theta)\, v_1 + (c_1\, \sin\theta + c_2\, \cos\theta)\, v_2.$$

We shall often denote $A(g,\, (m,\, v))$ by $g(m,\, v)$. Then $g : S(M) \to S(M)$ is a smooth map for each $g \in S^1$.

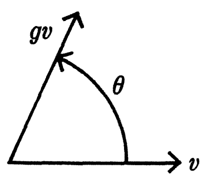

Figure 7.1

(3) If $\pi : S(M) \to M$ denotes projection, then $\pi^{-1}(m)$ is just the unit circle in $T(M,\, m)$. Moreover, if $(m,\, v_1)$ and $(m,\, v_2)$ are any two elements of $\pi^{-1}(m)$, then there exists a unique $g \in S^1$ such that $(m,\, v_2) = g(m,\, v_1)$ (Take $g = e^{i\theta}$, where θ is the positive angle of rotation from v_1 to v_2.)

(4) $S(M)$ is locally a product space. For let U be a coordinate neighborhood in M, with coordinate functions $(x_1,\, x_2)$. Let e_1 be the vector field $(\partial/\partial x_1)/\|\partial/\partial x_1\|$, where $\|\partial/\partial x_1\| = \langle \partial/\partial x_1,\, \partial/\partial x_1\rangle^{1/2}$. Then e_1 is a smooth vector field on U, which is everywhere of length 1. Thus e_1 defines a smooth map

$$c : U \to \pi^{-1}(U) \quad \text{by} \quad c(m) = (m,\, e_1(m)).$$

Clearly $\pi \circ c = i_U$. Now define $B : U \times S^1 \to \pi^{-1}(U)$ by

$$B(m,\, g) = gc(m) = (m,\, ge_1(m)) = A(g,\, (m,\, e_1(m))).$$

Then it is easy to verify that B is smooth, injective, and surjective; and that dB is everywhere nonsingular so that B^{-1} is also smooth.

(5) It is not true that $S(M)$ is globally a product of S^1 with M. If there exists a smooth nonzero vector field on M, then the above argument shows that $S(M)$ is diffeomorphic with $M \times S^1$. However, there do not exist such nonzero vector fields in general. (For example, $M = S^2$.)

For $M = \mathbb{R}^2$, the notion of translating a: tangent vector parallel to itself is clear. We now propose to generalize it and introduce the concept of *parallel translation* of tangent vectors on arbitrary 2-dimensional oriented Riemannian manifolds. It will tum out that we will be able to parallel translate vectors along curves from one point to another, but that the result will depend on the curve. In particular, if we parallel translate around a closed curve, we may not get back to our original vector. The new vector will differ from the original vector by a rotation; i.e., by an element of S^1. For $M = \mathbb{R}^2$, a "flat" space, this rotation is zero. For arbitrary M, this rotation (or, more precisely, the limit of it as the curve shrinks to a point m) will measure the "curvature" of M at m.

We shall require that parallel translation be an isometry. Thus, parallel translation of a unit vector along a curve $\alpha : [a,\, b] \to M$ will determine a unit tangent vector $\tilde{\alpha}(t) \in T(M,\, \alpha(t))$ for each $t \in [a,\, b]$. If $v \in \pi^{-1}(\alpha(a))$, then parallel translation of v will determine a curve $\tilde{\alpha} : [a,\, b] \to S(M)$ such that $\pi \circ \tilde{\alpha} = \alpha$. Moreover,

$$\text{if } v_1 \in \pi^{-1}(\alpha(a)) \quad \text{and} \quad v_1 = gv \quad \text{for some } g \in S^1, \text{ then}$$

the curve $\tilde{\alpha}_1 : [a,\, b] \to S(M)$ determined by parallel translating v_1 will be given by

$$\tilde{\alpha}_1(t) = g\tilde{\alpha}(t) \quad \text{for each } t \in [a,\, b].$$

Conversely, if, corresponding to each curve $\alpha : [a,\, b] \to M$ and each unit tangent vector v at $\alpha(a)$, there existed a unique "lift" $\tilde{\alpha} : [a,\, b] \to S(M)$, with the above properties, then a notion of parallel translation is defined (see Figure 7.2).

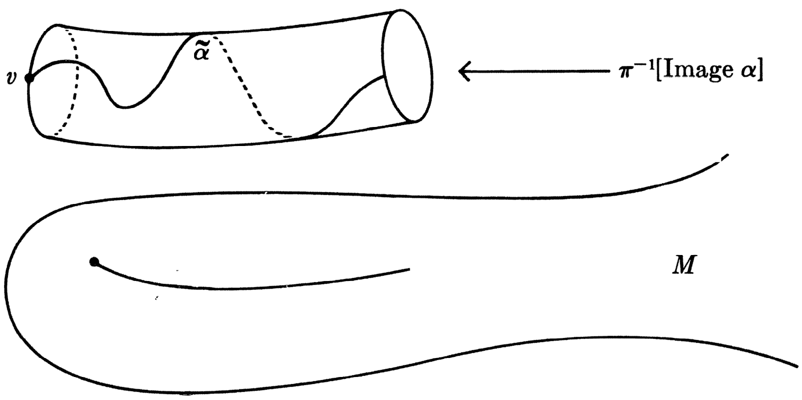

Figure 7.2

Recall that in the theory of covering spaces, each curve had a unique lift because the fibers $p^{-1}(x)$ were discrete. However, here the fibers $\pi^{-1}(m)$ are not discrete; they are circles. Hence lifts are not unique. In fact, we do not even know in which direction to start moving. (There is a whole line of vectors $\tilde{v} \in T(S(M), (m, v))$ such that $d\pi(\tilde{v}) = \dot{\alpha}(a)$; each of these is a candidate for $\dot{\tilde{\alpha}}(a)$.) So given $m \in M$ and $v \in T(M, m)$, we need a way of determining, for each curve α through m, an initial direction for $\tilde{\alpha}$; that is, we need a way of choosing, for each $\dot{\alpha}(a) \in T(M, m)$, a vector $\dot{\tilde{\alpha}}(a) \in T(S(M), (m, v))$ such that $d\pi(\dot{\tilde{\alpha}}(a)) = \dot{\alpha}(a)$. Choosing the vector $\dot{\tilde{\alpha}}(a)$ is more primitive than finding the lift $\tilde{\alpha}$, but it will turn out that when the choice is made at every point of $\pi^{-1}([a, b])$, the lift (hence the parallel translate) is determined.

A natural way of uniquely determining such a vector $\dot{\tilde{\alpha}}(a)$ would be to require that it lie in a given two-dimensional subspace of $T(S(M), (m, v))$ that is mapped isomorphically onto $T(M, m)$ by $d\pi : T(S(M), (m, v)) \to T(M, m)$. Such a subspace will be complementary to the *vertical space*

$$d\pi^{-1}(0) = \{t \in T(S(M), (m, v)) \mid d\pi(t) = 0\}.$$

Definition A *connection* on $S(M)$ is a choice of a two-dimensional subspace $\mathscr{H}(m, v)$ of $T(S(M), (m, v))$ at each point $(m, v) \in S(M)$ such that the following hold.

(1) $T(S(M), (m, v)) = \mathscr{H}(m, v) \bigoplus d\pi^{-1}(0)$; that is, the subspace $\mathscr{H}(m, v)$ is complementary to the vertical space at (m, v).

(2) $dg\left(\mathscr{H}(m, v)\right) = \mathscr{H}(m, gv)$ for each $g \in S^1$.

(3) The choice of $\mathscr{H}$ is smooth; that is, for each point $(m, v) \in S(M)$, there exists an open set U about (m, v) and smooth vector fields X and Y defined on U such that $\{X, Y\}$ spans $\mathscr{H}$ at each point of U.

Remark For a connection $\mathscr{H}$ on $S(M)$ there is an associated 1-form called the connection 1-form. To define it, we will construct a smooth vector field V on $S(M)$ which spans $d\pi^{-1}(0)$ at every point in $S(M)$. Let $\partial/\partial\theta$ denote the usual unit tangent vector field on S^1. Then $\partial/\partial\theta$ is invariant under the action of any $g \in S^1$ on S^1, since

$$dg\left(\frac{\partial}{\partial\theta}\right)\bigg|_h = \frac{\partial}{\partial\theta}\bigg|_{gh} \quad \text{for every } h \in S^1.$$

For each $(m, v) \in S(M)$, consider the smooth map $G : S^1 \to S(M)$ defined by

$$G(g) = g(m, v) = (m, gv).$$

Define $V : S(M) \to T(S(M), (m, v))$ by

$$V(m, v) = dG\left(\frac{\partial}{\partial\theta}\bigg|_1\right), \quad \text{where 1 is the unit in } S^1 \quad \text{(see Figure 7.3).}$$

In terms of a local coordinate neighborhood U of m in M and of the corresponding direct sum representation

$$T(S(M), (m, v)) = T(M, m) \bigoplus T(S^1, g) \quad ((m, v) \in \pi^{-1}(U)),$$

where g is such that $v = ge_1$ the vector field V is given by

$$V(m, v) = \left(0, \frac{\partial}{\partial\theta}\bigg|_g\right).$$

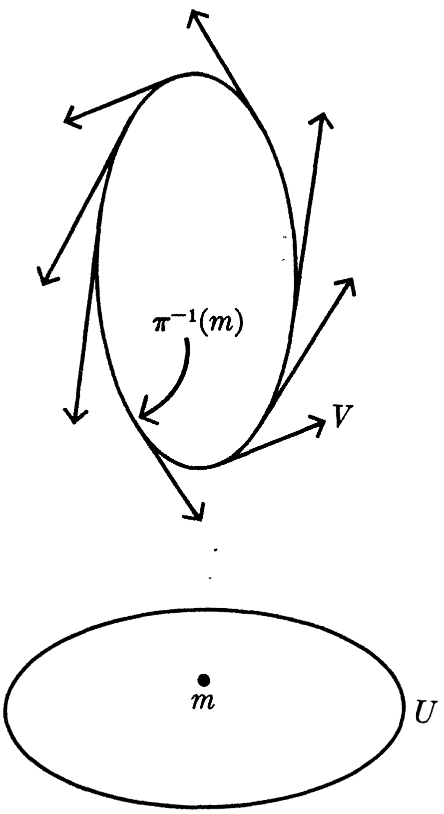

Figure 7.3

In particular, note that V is smooth and never zero, that $d\pi(V) = 0$, and that $dh(V) = V$ for every $h \in S^1$.

Definition Let $\mathscr{H}$ be a connection on $S(M)$. The 1-form of $\mathscr{H}$, or the connection 1-form, is the 1-form φ on $S(M)$ defined as follows. Let

$$q : T(S(M), (m, v)) = \mathscr{H}(m, v) \bigoplus d\pi^{-1}(0) \to d\pi^{-1}(0)$$

be the projection map. For $t \in T(S(M), (m, v))$, set $\varphi(t) = \lambda$, where λ is the real number such that $q(t) = \lambda V(m, v)$.

Local description of φ. Let X and Y be smooth vector fields defined in an open set U of $S(M)$ such that $\{X(m, v), Y(m, v)\}$ spans $\mathscr{H}(m, v)$ for each $(m, v) \in U$. Then $\{V(m, v), X(m, v), Y(m, v)\}$ is a basis for $T(S(M), (m, v))$ at each $(m, v) \in U$. Let $\{\varphi_1(m, v), \varphi_2(m, v), \varphi_3(m, v)\}$ be the dual basis for $T^*(S(M), (m, v))$. hen $\varphi_1, \varphi_2, \varphi_3$ are smooth 1-forms on U, and $\varphi = \varphi_1$. In particular,

(1) φ is smooth, since φ_1 is smooth.

(2) $\varphi(V) \equiv 1$.

(3) $g^*\varphi = \varphi$ for each $g \in S^1$. For if $t \in T(S(M), (m, v))$, then

$$t = \lambda V + t_1 \quad (\lambda \in \mathbb{R};\, t_1 \in \mathscr{H}),$$

and

$$\begin{aligned} (g^*\varphi)(t) &= \varphi \circ dg(t) = \varphi(\lambda\, dg(V) + dg(t_1)) = \varphi(\lambda V) \quad \text{since } dg(\mathscr{H}) \subset \mathscr{H} \\ &= \lambda = \varphi(t) \end{aligned}$$

Lemma Suppose ψ is any smooth 1-form on $S(M)$ such that $\psi(V) \equiv 1$ and $g^*\psi = \psi$. Then $\mathscr{H} = \psi^{-1}(0)$ is a connection on $S(M)$ with the property that its connection 1-form is ψ.

Proof For each $(m, v) \in S(M)$, $\psi(m, v) : T(S(M), (m, v)) \to \mathbb{R}$ is a linear functional. Since $\dim T(S(M), (m, v)) = 3$, $\psi^{-1}(0)$ has dimension 2. $V \notin \psi^{-1}(0)$, so $\psi^{-1}(0)$ is a complement to the vertical space. $dg\left(\psi^{-1}(0)\right) = \psi^{-1}(0)$ because $g^*\psi = \psi$.

Remark Let U be a coordinate neighborhood in M. We now exhibit a connection on $\pi^{-1}(U) = S(U) = U \times S^1$. Recall that, given coordinates (x_1, x_2) in U, a smooth map $c : U \to \pi^{-1}(U)$ is defined by

$$c(m) = (m, (\partial/\partial x_1)/\|\partial/\partial x_1\|).$$

For $m \in M$, let

$$\mathscr{H}_1(c(m)) = dc(T(U, m)).$$

Then $\mathscr{H}_1(c(m))$ is complementary to the vertical. For

$$d\pi\left(\mathscr{H}_1(c(m))\right) = d\pi \circ dc(T(U, m)) = d(\pi \circ c)(T(U, m)) = (T(U, m)).$$

so that $\mathscr{H}_1(c(m))$ is two-dimensional and $d\pi|_{\mathscr{H}_1(c(m))}$ is an isomorphism. Furthermore, $V \notin \mathscr{H}$ since $d\pi(V) = 0$.

Now set $\mathscr{H}_1(gc(m)) = dg\left(\mathscr{H}_1(c(m))\right)$.

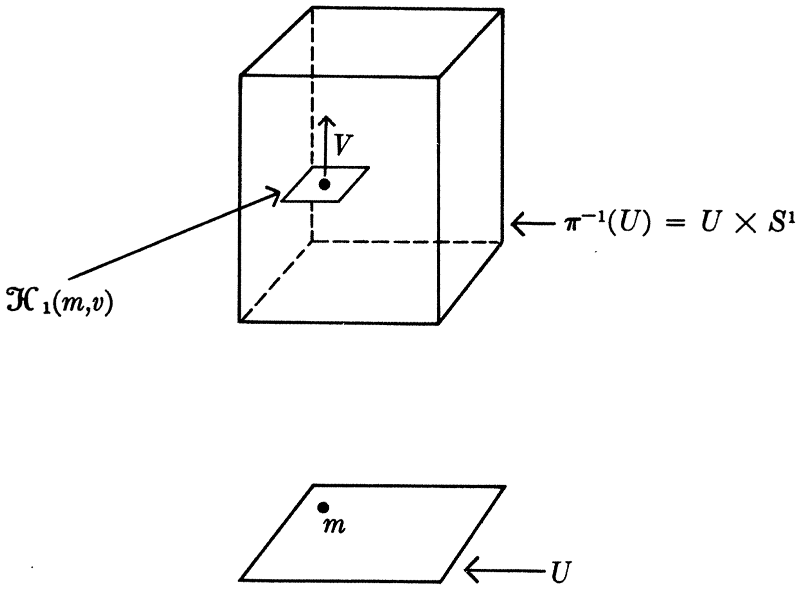

Figure 7.4

In terms of the product representation $\pi^{-1}(U) = U \times S^1$ given by c, $\mathscr{H}_1(m, v)$ is just the tangent space at (m, v) to the submanifold $U \times \{v\}$ (Figure 7.4). More precisely, letting $B : U \times S^1 \to \pi^{-1}(U)$ be the isomorphism defined by $B(m, g) = gc(m) = (m, ge_1(m))$,

$$\mathscr{H}_1(m, v) = dB\left(T(U \times \{g\}, (m, g))\right),$$

where $g \in S^1$ is such that $ge_1(m) = v$. The 1-form φ_1 of this connection is

$$\varphi_1 = \left(B^{-1}\right)^*\left(\widetilde{d\theta}\right),$$

where $p : U \times S^1 \to S^1$ is projection, $d\theta$ is the 1-form on S^1 dual to $\partial/\partial\theta$, and

$$\widetilde{d\theta} = p^*(d\theta).$$

Note that $d\varphi_1 = 0$ for this special connection, for

$$d\varphi_1 = d\left[\left(B^{-1}\right)^* \circ p^*(d\theta)\right] = d\left[\left(p \circ B^{-1}\right)^* (d\theta)\right] = \left(p \circ B^{-1}\right)^* (d(d\theta)) = 0.$$

Warning $d(d\theta) = 0$, not because $d\theta$ is the differential of a 0-form (it is not), but because there are no nonzero 2-forms on S^1.

Our definition of a connection was motivated by a desire to construct a notion of parallel translation. We now prove that given a connection on $S(M)$, parallel translation is indeed defined.

Theorem Let $\mathscr{H}$ be a connection on $S(M)$ with 1-form φ. Let $\alpha : [a, b] \to M$ be a broken C^∞ curve in M. Let $v \in T(M, \alpha(a))$ with $\|v\| = 1$. Then there exists a unique broken C^∞ curve $\tilde{\alpha} : [a, b] \to S(M)$, called the *horizontal lift* of α, through $(\alpha(a), v)$, such that

(1) $\pi \circ \tilde{\alpha} = \alpha$.

(2) $\dot{\tilde{\alpha}}(t) \in \mathscr{H}(\tilde{\alpha}(t))$; that is, $\varphi\left(\dot{\tilde{\alpha}}(t)\right) = 0$ for all $t \in [a, b]$.

(3) $\tilde{\alpha}(a) = (\alpha(a), v)$.

The vector $\tilde{\alpha}(b) \in T(M, \alpha(b))$ is the parallel translate of v along α to $\alpha(b)$.

The proof of this theorem requires two preliminary lemmas.

Lemma 1 Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two connections on $S(M)$ with connection 1-forms φ_1 and φ_2. Then

(1) $(\varphi_2 - \varphi_1)(V) = 0$.

(2) $g^*(\varphi_2 - \varphi_1) = \varphi_2 - \varphi_1$ for all $g \in S^1$.

(3) $\varphi_2 - \varphi_1 = \pi^*(\tau)$ for some smooth 1-form τ on M.

Proof (1) and (2) are clear. We shall show that (1) and (2) imply (3). If ψ is any smooth 1-form on $S(M)$ with $\psi(V) \equiv 0$ and $g^*(\psi) = \psi$ for all $g \in S^1$, then $\psi = \pi^*(\tau)$ for some τ. To define τ on $v \in T(M, m)$, let $(m, v_1) \in \pi^{-1}(m)$, and let $w \in T(S(M), (m, v_1))$, be such that $d\pi(w) = v$. Set $\tau(v) = \psi(w)$. $\tau(v)$ is independent of the w chosen in $d\pi^{-1}(v)$ since $d\pi(w_1) = v$ implies that $d\pi(w_1 - w) = 0$, so that $w_1 - w = \lambda V$ for some λ. Thus

$$\psi(w_1) = \psi(w + \lambda V) = \psi(w) + \lambda\,\psi(V) = \psi(w).$$

Also, $\tau(v)$ is independent of the point (m, v_1) chosen in $\pi^{-1}(m)$, because if

$$(m, v_2) \in \pi^{-1}(m),$$

then $v_2 = gv_1$ for some $g \in S^1$. Moreover, if $w \in T(S(M), (m, v_1))$, satisfies $d\pi(w) = v$, then $dg(w) \in T(S(M), (m, v_2))$, satisfies $dw(dg(w)) = v$, and

$$\psi|_{(m, v_2)}(dg(w)) = \psi|_{g(m, v_1)}(dg(w)) = g^*\psi|_{(m, v_1)}(w) = \psi|_{(m, v_1)}(w).$$

τ is smooth because in a coordinate neighborhood U, $\tau(v) = \psi \circ dc(v)$, where

$$c : U \to \pi^{-1}(U)$$

is defined by $c(m) = (m, e_1(m))$.

Lemma 2 Let $\alpha : [a, b] \to M$ be a smooth curve in M. Let $\tilde{\alpha} : [a, b] \to S(M)$ and $\tilde{\beta} : [a, b] \to S(M)$ be smooth curves such that $\pi \circ \tilde{\alpha} = \alpha$ and $\pi \circ \tilde{\beta} = \alpha$ (see Figure 7.5). Suppose $\tilde{\alpha}$ is horizontal relative to some connection $\mathscr{H}$ on $S(M)$ with connection 1-form φ; that is, suppose $\varphi\left(\dot{\tilde{\alpha}}(t)\right) = 0$. Then there exists a smooth function $\theta : [a, b] \to \mathbb{R}$ such that

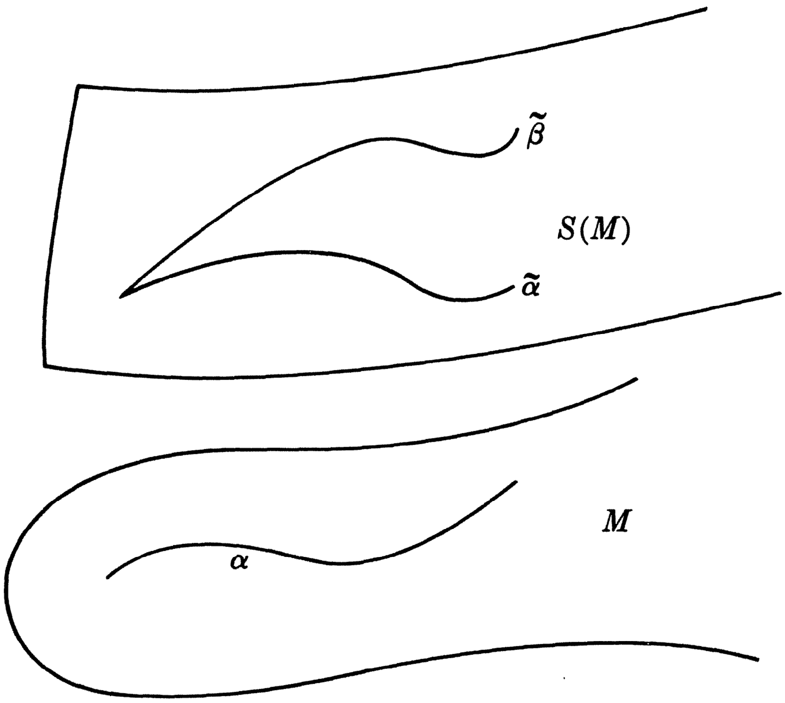

Figure 7.5

(1) $\tilde{\beta}(t) = e^{i\theta(t)}\tilde{\alpha}(t)$ for $t \in [a,\, b]$ and

(2) $\varphi\left(\dot{\tilde{\beta}}(t)\right) = (d\theta/dt)\,(t)$ for $t \in [a,\, b]$.

Furthermore, if $\tilde{\alpha}(a) = \tilde{\beta}(a)$, then θ can be chosen such that $\theta(a) = 0$.

Proof Let $\hat{g} : [a,\, b] \to S^1$ be defined by

$$\tilde{\beta}(t) = \hat{g}(t)\,\tilde{\alpha}(t) \quad \text{for } t \in [a,\, b].$$

It is easy to verify that $\hat{g}$ is a smooth curve. Since $\mathbb{R}$ is a covering space of S^1, and $[a,\, b]$ is simply connected, there exists a lift $\theta : [a,\, b] \to \mathbb{R}$ of $\hat{g}$ (see Figure 7.6). Furthermore, if $\tilde{\alpha}(a) = \tilde{\beta}(a)$, then $\hat{g}(a) = 1$, and there exists a unique such lift with $\theta(a) = 0$.

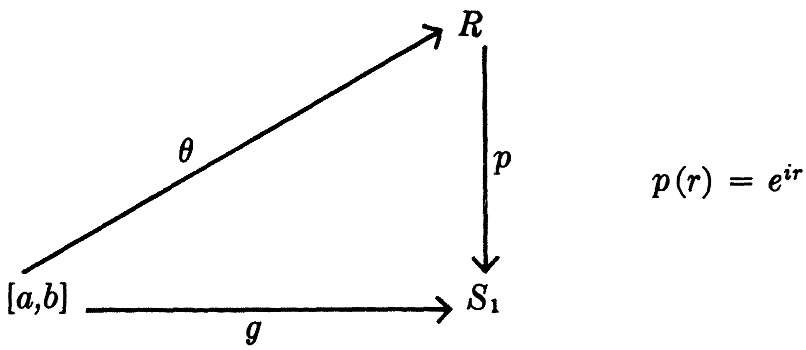

Figure 7.6

Since p is smooth and has a smooth inverse locally, θ is smooth. Furthermore,

$$\tilde{\beta}(t) = \hat{g}(t)\,\tilde{\alpha}(t) = p \circ\, \theta(t)\,\tilde{\alpha}(t) = e^{i\theta(t)}\,\tilde{\alpha}(t)$$

so (1) is satisfied.

To verify (2), first note that the tangent vector to the curve

$$\hat{g} : [a,\, b] \to S^1 \quad (\hat{g}(t) = e^{i\theta(t)})$$

is given by

$$\dot{\hat{g}}(t) \;=\; (p \dot{\circ} \theta)\,(t) = d\,(p \circ \theta)\left(\frac{d}{dt}\right) = dp\left(d\theta\left(\frac{d}{dt}\right)\right) = dp\left(\frac{d\theta}{dt}\,\frac{d}{dt}\right) = \frac{d\theta}{dt}\,dp\left(\frac{d}{dt}\right) = \frac{d\theta}{dt}\,\frac{\partial}{\partial\theta}$$

Restricting attention to a coordinate neighborhood $U \subset M$ and the corresponding product representation $\pi^{-1}(U) \cong U \times S^1$,

$$\tilde{\alpha}(t) = h(t)\, c(\alpha(t)) = (\alpha(t),\, h(t)),$$

for some $h(t) = e^{i\psi(t)} \in S^1$, and

$$\tilde{\beta}(t) = (\alpha(t),\, \hat{g}(t)\, h(t)) = (\alpha(t),\, e^{i(\theta(t)+\psi(t))}).$$

The tangent vector to $\tilde{\alpha}$ at $\tilde{\alpha}$ is then $(\dot{\alpha}(t),\, (d\psi/dt)\,(\partial/\partial\theta))$, whereas the tangent vector to $\tilde{\beta}$ at $\tilde{\beta}(t)$ is then $(\dot{\alpha}(t),\, [(d\theta/dt) + (d\psi/dt)]\; (\partial/\partial\theta))$,; that is,

$$\dot{\tilde{\beta}}(t) = d\,(\hat{g}(t))\;\left(\dot{\tilde{\alpha}}(t)\right) + \left(0,\, \frac{d\theta}{dt}\,\frac{\partial}{\partial\theta}\right) = d\,(\hat{g}(t))\;\left(\dot{\tilde{\alpha}}(t)\right) + \frac{\partial\theta}{dt}\,V,$$

where $d\,(\hat{g}(t))$ is the differential of the map $\hat{g} : S(M) \to S(M)$. Since $\dot{\tilde{\alpha}}(t)$ is horizontal, and $d\,(\hat{g}(t))\,(\mathscr{H}) \subset \mathscr{H}$,

$$\varphi\left(\dot{\tilde{\beta}}(t)\right) = \frac{d\theta}{dt}\,\varphi(V) = \frac{d\theta}{dt}.$$

Proof of the Theorem Note that it suffices to prove the theorem for a smooth curve α. For then we can uniquely lift each smooth portion of any broken curve.

Local existence Let U be a coordinate neighborhood in M. We shall show the existence of unique horizontal lifts in $\pi^{-1}(U) = S(U)$. Let $c : U \to S(U)$ be as usual: $c(m) = (m,\, e_1(m))$ for $m \in U$. We shall first show that if $\mathscr{H}$ is the special connection $\mathscr{H}_1$ on $\pi^{-1}(U)$, constructed via the product structure, then α has a unique horizontal lift $\tilde{\alpha}_1$ such that $\tilde{\alpha}_1(a) = c(\alpha(a))$. Indeed, let $\tilde{\alpha}_1 : [a,\, b] \to \pi^{-1}(U)$ be defined by $\tilde{\alpha}_1 = c \circ \alpha$. Then $\pi \circ \tilde{\alpha}_1 = \pi \circ c \circ \alpha = \alpha$ and $\dot{\tilde{\alpha}}_1(t) = dc\,(\dot{\alpha}(t)) \in \mathscr{H}_1\,(c(t))$, so $\tilde{\alpha}_1$ is a horizontal lift. Moreover, $\tilde{\alpha}_1$ is the unique $\mathscr{H}_1$-horizontal lift such that

$$\tilde{\alpha}_1(a) = c(\alpha(a)).$$

For if $\tilde{\alpha}_2$ were another such lift, then, by Lemma 2,

$$\tilde{\alpha}_2(t) = e^{i\theta(t)}\tilde{\alpha}_1(t)$$

for some smooth function θ with $\theta(a) = 0$; and $\varphi_1\left(\dot{\tilde{\alpha}}_2(t)\right) = d\theta/dt$, where φ_1 is the connection 1-form of $\mathscr{H}_1$. Now $\tilde{\alpha}_2$ is $\mathscr{H}_1$-horizontal if and only if $\varphi_1\left(\dot{\tilde{\alpha}}_2(t)\right) \equiv 0$; that is, $d\theta/dt \equiv 0$. Hence $\theta(t)$ must be constant. Since $\theta(a) = 0$, $\theta(t) \equiv 0$; that is, $\tilde{\alpha}_2(t) = \tilde{\alpha}_1(t)$ for all t; that is, $\tilde{\alpha}_2 = \tilde{\alpha}_1$.

Thus α admits a unique $\mathscr{H}_1$-horizontal lift $\tilde{\alpha}_1$ with $\tilde{\alpha}_1(a) = c(\alpha(a))$. Now consider our original connection with connection 1-form φ. Then, by Lemma 1,

$$\varphi_2 - \varphi_1 = \pi^*(\tau)$$

for some smooth 1-form τ on U. Let $\tilde{\alpha}$ be any curve in $\pi^{-1}(U)$ such that $\pi \circ \tilde{\alpha} = \alpha$. Then, by Lemma 2, $\tilde{\alpha}(t) = e^{i\theta(t)}\tilde{\alpha}_1(t)$ and $\varphi_1\left(\dot{\tilde{\alpha}}(t)\right) = d\theta/dt$. Thus $\tilde{\alpha}$ is an $\mathscr{H}$-horizontal lift of α if and only if $\varphi\left(\dot{\tilde{\alpha}}(t)\right) \equiv 0$; that is, if and only if

$$(\varphi_1 - \varphi)\left(\dot{\tilde{\alpha}}(t)\right) \equiv \varphi_1\left(\dot{\tilde{\alpha}}(t)\right) = \frac{d\theta}{dt}.$$

But on the other hand,

$$(\varphi_1 - \varphi)\left(\dot{\tilde{\alpha}}(t)\right) = (\pi^*\tau)\left(\dot{\tilde{\alpha}}(t)\right) = \tau\left(d\pi\dot{\tilde{\alpha}}(t)\right) = \tau\left(\dot{\alpha}(t)\right).$$

Thus α is $\mathscr{H}$-horizontal if and only if $d\theta/dt = \tau\left(\dot{\alpha}(t)\right)$; that is, $\theta = \displaystyle\int_0^t \tau\left(\dot{\alpha}(t)\right)\, dt + \theta_0$ for some constant θ_0. Hence each $\mathscr{H}$-horizontal lift $\tilde{\alpha}$ of α; is of the form

$$\tilde{\alpha}(t) = \hat{g}(t)\left(c \circ \alpha(t)\right),$$

where

$$\hat{g}(t) = e^{i\theta_0}\, e^{i\int_0^t \tau(\dot{\alpha}(t))\, dt}$$

For each unit vector v in $T(U, \alpha(a))$, there is precisely one θ_0 with $0 \le \theta_0 < 2\pi$ and $(\alpha(a), v) = e^{i\theta_0}\left(\alpha(a), e_1\right)$. The above formula, with this value of θ_0, then gives the unique $\mathscr{H}$-horizontal lift $\tilde{\alpha}$ with $\tilde{\alpha}(a) = (\alpha(a), v)$.

Global existence To establish global existence, let $\alpha : [a, b] \to M$ and let

$$t_0 = \sup\{t \in [a, b] \mid \alpha|_{[a,b]} \text{ has a (unique) lift } \tilde{\alpha}\}.$$

We shall show that $t_0 = b$. Suppose $t_0 \ne b$. Then consider the restriction of α to the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$. By local existence, this has a unique lift $\tilde{\tilde{\alpha}}$ for some sufficiently small $\varepsilon > 0$, say with $\tilde{\tilde{\alpha}}(t_0) = (\alpha(t_0), w) \in S(M)$. Then $\tilde{\alpha}(t_0 - \varepsilon) = g\tilde{\tilde{\alpha}}(t_0 - \varepsilon)$ for some $g \in S^1$, and $g\tilde{\tilde{\alpha}}$ is a horizontal lift with $g\tilde{\tilde{\alpha}}(t_0 - \varepsilon) = \tilde{\alpha}(t_0 - \varepsilon)$. By uniqueness, $g\tilde{\tilde{\alpha}} = \tilde{\alpha}$ on the interval $[t_0 - \varepsilon, t_0)$. Hence $g\tilde{\tilde{\alpha}}$ extends $\tilde{\alpha}$ beyond t_0, contradicting the definition of t_0.

Remark Note that, relative to the special connection $\mathscr{H}_1$ on $\pi^{-1}(U)$, parallel translation is independent of the curve. In fact, the vector field $e_1 = (\partial/\partial x_1)\,/\,\|\partial/\partial x_1\|$ is parallel along every curve in U.

§7.2 Structural Equations and Curvature

Definition Consider the circle bundle $S(M)$ of a smooth oriented Riemannian 2-manifold M. Two smooth 1-forms ω_1 and ω_2 are defined on $S(M)$ as follows. For $t \in T(S(M), (m, v))$,

$$\begin{aligned} \omega_1(t) &= \langle d\pi(t), v\rangle, \\ \omega_2(t) &= \langle d\pi(t), iv\rangle, \end{aligned}$$

where $iv = e^{i\pi/2}v$ is the image of v under rotation through an angle of $\pi/2$ in $T(M, m)$. (We shall show below that these 1-forms are indeed smooth.)

Remark 1 Thus $\omega_1(t)$ and $\omega_2(t)$ are the components of $d\pi(t)$ relative to the orthonormal basis $\{v, iv\}$ for $T(M, m)$; that is,

$$d\pi(t) = \omega_1(t)v + \omega_2(t)(iv).$$

Remark 2 Suppose $\mathscr{H}$ is a connection on $S(M)$ with connection 1-form φ. Then $\{\varphi, v, iv\}$ is a basis for $T^*(S(M), (m, v))$ for each $(m, v) \in S(M)$.

Proof Note that if $t \in T(S(M), (m, v))$ such that $\omega_1(t) = \omega_2(t) = 0$, then $d\pi(t) = 0$, so t is vertical; that is, $t = \lambda V$ for some λ. Furthermore, if $\varphi(t) = 0$, then $\lambda = \varphi(\lambda V) = \varphi(t) = 0$, so $t = 0$. Thus $\{\varphi, v, iv\}$ are linearly independent and, since $\dim T^*(S(M), (m, v)) = 3$, form a basis for $T^*(S(M), (m, v))$ for each $(m, v) \in S(M)$.

Remark 3 Let $g = e^{i\theta} \in S^1$. Then

$$\begin{aligned} g^*\omega_1 &= (\cos\theta)\,\omega_1 + (\sin\theta)\,\omega_2 \\ g^*\omega_2 &= -(\sin\theta)\,\omega_1 + (\cos\theta)\,\omega_2. \end{aligned}$$

Proof $gv = (\cos\theta)\,v + (\sin\theta)\,(iv)$. Hence, for $t \in T(S(M), (m, v))$,

$$\begin{aligned} g^*\omega_1|_{(m,v)}(t) &= \omega_1|_{(m,gv)}(dg(t)) \\ &= \langle d\pi \circ dg(t), gv\rangle \\ &= \langle d\pi(t), gv\rangle \\ &= \langle d\pi(t), (\cos\theta)\,v + (\sin\theta)\,iv\rangle \\ &= (\cos\theta)\,\omega_1 + (\sin\theta)\,\omega_2. \end{aligned}$$

Similarly,

$$\begin{aligned} g^*\omega_2|_{(m,v)}(t) &= \omega_2|_{(m,gv)}(dg(t)) \\ &= \langle d\pi \circ dg(t), igv\rangle \\ &= \langle d\pi(t), igv\rangle \\ &= \langle d\pi(t), (\cos\theta)\,iv - (\sin\theta)\,v\rangle \\ &= -(\sin\theta)\,\omega_1 + (\cos\theta)\,\omega_2. \end{aligned}$$

Remark 4 $g^*(\omega_1 \wedge \omega_2) = \omega_1 \wedge \omega_2$ for all $g = e^{i\theta} \in S^1$. For,

$$g^*(\omega_1 \wedge \omega_2) = g^*\omega_1 \wedge g^*\omega_2 = \left(\cos^2\theta + \sin^2\theta\right)\omega_1 \wedge \omega_2 = \omega_1 \wedge \omega_2.$$

Furthermore, $\omega_1 \wedge \omega_2\,(t_1, t_2) = 0$ if either t_1 or t_2 is vertical. Hence, as in the proof that $\varphi - \varphi_1 = \pi^*\tau$ for some τ (Section 7.1), the 2-form $\omega_1 \wedge \omega_2$ is the image under π^* of a (unique) form on M.

Definition The *volume element* of a smooth oriented Riemannian 2-manifold M is the smooth 2-form, vol, on M such that

$$\pi^*(\mathrm{vol}) = \omega_1 \wedge \omega_2;$$

that is, for $v_1, v_2 \in T(M, m)$, $\mathrm{vol}(v_1, v_2) = \omega_1 \wedge \omega_2|_{(m,v)}(v_1', v_2')$ for any

$$(m, v) \in \pi^{-1}(m) \subset S(M) \quad \text{and} \quad v_1', v_2' \in T(S(M), (m, v))$$

such that $d\pi(v_i') = v_i$ for $i = 1, 2$.

Remark 5 Suppose U is a coordinate neighborhood in M with coordinate functions (x_1, x_2). Let $e_1 = (\partial/\partial x_1)/\|\partial/\partial x_1\|$ and let ω_1', ω_2' be the smooth 1-forms on U that at each $m \in U$ form the basis for $T^*(M, m)$ dual to $\{e_1(m), ie_1(m)\}$. Let

$$c : U \to \pi^{-1}(U) \subset S(M)$$

be given by $c(m) = (m, e_1(m))$. Then, for $v \in T(M, m)$,

$$(c^*\omega_1)(v) = \omega_1(dc(v)) = \langle d\pi \circ dc(v), e_1\rangle = \langle v, e_1\rangle = \omega_1'(v),$$

so $\omega_1' = c^*\omega_1$. Similarly, $\omega_2' = c^*\omega_2$. In particular,

$$\omega_1' \wedge \omega_2' = c^*\omega_1 \wedge c^*\omega_2 = c^*(\omega_1 \wedge \omega_2) = c^* \circ \pi^*(\mathrm{vol}) = (c \circ \pi)^*(\mathrm{vol});$$

so, since $\pi \circ c = i_U$,

$$\boxed{\mathrm{vol}|_U = \omega_1' \wedge \omega_2'}$$

Now let $\tilde{\omega}_i = \pi^*\omega_i'$ $(i = 1, 2)$. Then $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are smooth 1-forms on $\pi^{-1}(U) \subset S(M)$ and

$$\tilde{\omega}_1 \wedge \tilde{\omega}_2 = \pi^*\omega_1' \wedge \pi^*\omega_2' = \pi^*(\omega_1' \wedge \omega_2') = \pi^*(\mathrm{vol}) = \omega_1 \wedge \omega_2.$$

Moreover, at each point $(m, e_1(m))$ of $C(U)$, $\omega_i = \tilde{\omega}_i$. For; if

$$t \in T(S(M), (m, e_1))$$

then

$$\begin{aligned} \omega_1(t)\, e_1 + \omega_1(t)\,(ie_1) &= d\pi(t) \\ &= \omega_1'(d\pi(t))\, e_1 + \omega_2'(d\pi(t))\,(ie_1) \\ &= \tilde{\omega}_1(t)\, e_1 + \tilde{\omega}_2(t)\,(ie_1) \end{aligned}$$

Note further that, for $g = e^{i\theta} \in S^1$,

$$g^*\tilde{\omega}_i = g^* \circ \pi^*\omega_i' = (\pi \circ g)^*\,\omega_i' = \pi^*\omega_i' = \tilde{\omega}_i.$$

Thus, from Remark 3 above,

$$\begin{aligned} (g^*\omega_1)|_{(m, e_1)} &= (\cos\theta)\,\omega_1 + (\sin\theta)\,\omega_2|_{(m, e_1)} \\ &= (\cos\theta)\,\tilde{\omega}_1 + (\sin\theta)\,\tilde{\omega}_2|_{(m, e_1)} \\ &= (\cos\theta)\, g^*\tilde{\omega}_1 + (\sin\theta)\, g^*\tilde{\omega}_2|_{(m, e_1)} \end{aligned}$$

Applying $\left(g^{-1}\right)^*$, the forms ω_1 and $\tilde{\omega}_i$ at (m, ge_1) are related by

$$\omega_1 = (\cos\theta)\,\tilde{\omega}_1 + (\sin\theta)\,\tilde{\omega}_2.$$

Similarly,

$$\omega_2 = -(\sin\theta)\,\tilde{\omega}_1 + (\cos\theta)\,\tilde{\omega}_2.$$

In particular, the above formulae show that ω_1 and ω_2 are smooth.

Remark 6 For higher dimensional Riemannian manifolds, the volume element is obtained similarly. If U is a coordinate neighborhood in the oriented Riemannian manifold M, with coordinate functions $(x_1, \dots, x_n)$ such that $dx_1 \wedge \cdots \wedge d_{x_n}$ gives the orientation of U, consider the vector fields $\partial/\partial x_1, \dots, \partial/\partial x_n$.

Using the Gram-Schmidt orthogonalization process, we obtain smooth vector fields $e_1, \dots, e_n$ on U which form an orthonormal basis for the tangent space at each point. Let $\omega_1', \dots, \omega_n'$ be the dual 1-forms. Then the n-form $\mathrm{vol}|_U = \omega_1' \wedge \cdots \wedge \omega_n'$ is independent of the (oriented) coordinate system on U and thus defines a global nonzero n-form vol.

Given an oriented Riemannian 2-manifold M and a connection on $S(M)$ with connection 1-form φ, the 1-forms $\varphi, \omega_1, \omega_2$ form a basis for the cotangent space at each point of $S(M)$. Hence the 2-forms $\omega_1 \wedge \omega_2$, $\omega_1 \wedge \varphi$, $\omega_2 \wedge \varphi$, form a basis for the 2-forms at each point of $S(M)$. Hence $d\varphi$, $d\omega_1$, $d\omega_2$ can be expressed in terms of this basis. The resulting formulae are called the *Cartan structural equations*. We now derive them, beginning with the second structural equation.

Second structural equation. On $\pi^{-1}(U)$, for a coordinate neighborhood U, let φ_1 denote the connection 1-form of the special connection $\mathscr{H}$. Then $d\varphi_1 = 0$ so that

$$d\varphi = d\varphi - d\varphi_1 = d\left(\varphi - \varphi_1\right) = d\left(\pi^*\tau\right) = \pi^*\left(d\tau\right)$$

for some smooth 1-form τ on U. Now $d\tau$ is a 2-form on U, hence is a multiple of the volume element; that is, $d\tau = -K\,\text{vol}$ for some smooth function K on U. Thus

$$d\varphi = \pi^*\left(-K\,\text{vol}\right) = \pi^*\left(-K\right)\pi^*\left(\text{vol}\right)$$

or

$$\boxed{d\varphi = -\left(K \circ \pi\right)\omega_1 \wedge \omega_2.}$$

The smooth function K is independent of the coordinates used, since it is determined by this last formula. Thus K is a smooth function on M, called the *curvature* of the connection φ.

First structural equation. On $\pi^{-1}(U)$, for a coordinate neighborhood U, we have seen that at $e^{i\theta}c(m)$,

$$\begin{aligned}\omega_1 &= \left(\cos\,\theta\right)\tilde{\omega}_1 + \left(\sin\,\theta\right)\tilde{\omega}_2,\\ \omega_2 &= -\left(\sin\,\theta\right)\tilde{\omega}_1 + \left(\cos\,\theta\right)\tilde{\omega}_2.\end{aligned}$$

Now

$$d\tilde{\omega}_i = d\left(\pi^*\omega_i'\right) = \pi^*\left(d\omega_i'\right) = \pi^*\left(a_i\,\text{vol}\right) = \left(a_i \circ \pi\right)\omega_1 \wedge \omega_2$$

for some smooth function a_i on U. Thus setting $\tilde{a}_i = a_i \circ \pi$,

$$\begin{aligned}d\omega_1 &= -\left(\sin\,\theta\right)\,d\theta \wedge \tilde{\omega}_1 + \left(\cos\,\theta\right)\,\tilde{a}_1\omega_1 \wedge \omega_2 + \cos\,\theta\,d\theta \wedge \tilde{\omega}_2 + \left(\sin\,\theta\right)\,\tilde{a}_2\omega_1 \wedge \omega_2\\ &= d\theta \wedge \omega_2 + \left(\tilde{a}_1\,\cos\,\theta + \tilde{a}_2\,\sin\,\theta\right)\omega_1 \wedge \omega_2\end{aligned}$$

If $\mathscr{H}$ is the special connection $\mathscr{H}_1$ on $\pi^{-1}(U)$, then $\varphi_1 = d\theta$, thus for this special connection,

$$(*) \qquad d\omega_1 = \varphi_1 \wedge \omega_2 + b_1\,\omega_1 \wedge \omega_2$$

for some smooth function b_1 on $\pi^{-1}(U)$. Similarly,

$$d\omega_2 = -\varphi_1 \wedge \omega_1 + b_2\,\omega_1 \wedge \omega_2$$

For an arbitrary connection form φ, $\varphi_1 - \varphi = \pi^*\tau$ for some smooth 1-form $\tau = c_1\omega_1' + c_2\omega_2'$ on U. Hence

$$\begin{aligned}\varphi_1 - \varphi &= \pi^*\left(c_1\omega_1' + c_2\omega_2'\right)\\ &= \left(c_1 \circ \pi\right)\tilde{\omega}_1 + \left(c_2 \circ \pi\right)\tilde{\omega}_2\\ &= f_1\,\omega_1 + f_2\,\omega_2\end{aligned}$$

for some smooth functions f_1, f_2 on $\pi^{-1}(U)$, since $\tilde{\omega}_1$, $\tilde{\omega}_2$ span the same space at each point as ω_1, ω_2. Thus

$$\varphi_1 = \varphi + f_1\,\omega_1 + f_2\,\omega_2,$$

and, by substituting into $(*)$,

$$\begin{aligned} d\omega_1 &= \varphi \wedge \omega_2 + f_1\,\omega_1 \wedge \omega_2 + b_1\,\omega_1 \wedge \omega_2 \\ &= \varphi \wedge \omega_2 + (f_1 + b_1)\,\omega_1 \wedge \omega_2. \end{aligned}$$

This, together with the corresponding equation for $d\omega_2$, gives the first structural equations as follows:

$$\boxed{\begin{aligned} d\omega_1 &= \varphi \wedge \omega_2 + h_1\,\omega_1 \wedge \omega_2, \\ d\omega_2 &= -\varphi \wedge \omega_1 + h_2\,\omega_1 \wedge \omega_2, \end{aligned}}$$

where h_1, h_2 are smooth functions on $S(M)$. Note that although these equations were derived over a coordinate neighborhood, they are independent of coordinates. Thus they are valid globally.

Although one might expect that by choosing an appropriate connection φ on $S(M)$, the coefficients of $d\omega_i$ relative to the basis $\{\varphi \wedge \omega_1,\ \varphi \wedge \omega_2,\ \omega_1 \wedge \omega_2\}$ could be prescribed fairly arbitrarily, this is not the case. In fact, $d\omega_1$ never has a component in the $\varphi \wedge \omega_1$ direction, and $d\omega_2$ never has a component in the $\varphi \wedge \omega_2$ direction. Moreover, the components of $d\omega_1$ and $d\omega_2$ in the $\varphi \wedge \omega_2$ and $\varphi \wedge \omega_1$ directions, respectively, must always be $+1$ and -1.

It is natural to ask whether the first structural equations can be made simpler by an appropriate choice of connection on $S(M)$. In particular, can φ be chosen such that $h_1 \equiv 0$ and $h_2 \equiv 0$? The answer is yes, and the choice is unique.

Theorem Let M be an oriented Riemannian 2-manifold. Then there exists a unique connection ψ on $S(M)$ such that

$$\boxed{\begin{aligned} d\omega_1 &= \psi \wedge \omega_2, \\ d\omega_2 &= -\psi \wedge \omega_1. \end{aligned}}$$

This connection is called the *Riemannian connection*.

Proof Let φ be any connection on $S(M)$. If ψ is any other connection on $S(M)$, then, as above,

$$\varphi - \psi = x_1\omega_1 + x_2\omega_2$$

for some x_1 and x_2. Solving for φ and substituting in the first structural equations for φ, we obtain

$$\begin{aligned} d\omega_1 &= \psi \wedge \omega_2 + (h_1 + x_1)\,\omega_1 \wedge \omega_2, \\ d\omega_2 &= -\psi \wedge \omega_1 + (h_2 + x_2)\,\omega_1 \wedge \omega_2, \end{aligned}$$

Thus

$$\begin{aligned} d\omega_1 &= \psi \wedge \omega_2 \\ d\omega_2 &= -\psi \wedge \omega_1 \end{aligned}$$

if and only if $x_1 = -h_1$, $x_2 = -h_2$. This gives both existence and uniqueness.

The Cartan structural equations have a dual formulation in terms of vector fields. Let V, E_1, E_2 be the smooth vector fields on $S(M)$ that form the dual basis to φ, ω_1, ω_2. Then E_1 and E_2 are horizontal at each point since $\varphi(E_1) = \varphi(E_2) = 0$. Moreover,

$$d\pi\,(E_1(m,\,v)) = \omega_1(E_1)v + \omega_2(E_1)(iv) = v,$$

so $E_1(m,\,v)$ is the unique horizontal vector at $(m,\,v)$ whose image under $d\pi$ is v. Similarly, $d\pi\,(E_2(m,\,v)) = iv$. The structural equations then become

$$\boxed{\begin{aligned}[V, E_1] &= E_2,\\ [V, E_2] &= -E_1,\\ [E_1, E_2] &= (K\circ\pi)\,V - h_1E_1 - h_2E_2.\end{aligned}}$$

If $\varphi = \psi$, the 1-form of the Riemannian connection, the last boxed equation reduces to

$$\boxed{[E_1, E_2] = (K\circ\pi)\,V.}$$

To verify these equations, apply the formula

$$d\tau\,(V_1, V_2) = \frac{1}{2}\{V_1\tau\,(V_2) - V_2\tau\,(V_1) - \tau\,([V_1, V_2])\}$$

nine times, as τ runs through the set $\{\varphi, \omega_1, \omega_2\}$, and V_1, V_2 runs through the set $\{V, E_1, E_2\}$.

Remark If K is constant, these formulae show that $\{V, E_1, E_2\}$ spans a finite-dimensional Lie algebra.

From now on, for an oriented Riemannian 2-manifold M, let the connection chosen be the Riemannian connection, and let K be the curvature function for that connection.

§7.3 Interpretation of Curvature

We now show that the curvature K of M measures the amount of rotation obtained in parallel translating vectors around small closed curves in M. The intuitive reason is this. On $S(M)$ we have the vector fields E_1, E_2 and V, and we know that for the Riemannian connection,

$$\boxed{[E_1, E_2] = (K\circ\pi)\,V.}$$

But $[E_1, E_2]\,(m, v)$ is just the tangent vector to the curve through (m, v) obtained by following the integral curves of E_1 and E_2 forward and then backward through parameter distances of $\sqrt{s}$. (Figure 7.7; see Section 5.3).

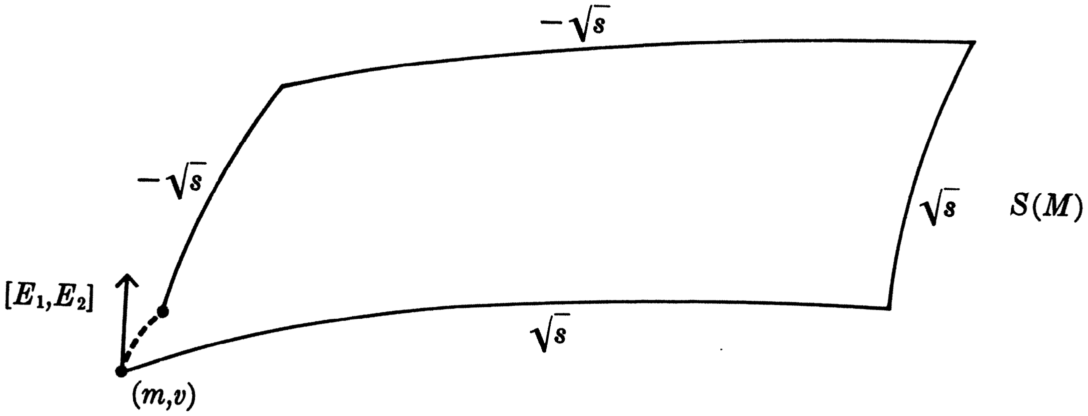

Figure 7.7

Projecting this figure down to M, we obtain a rectangular-shaped figure which is "nearly" closed; that is, the curve obtained through m has zero tangent vector at m because it is the projection of $[E_1, E_2]\,(m, v)$, which is vertical (see Figure 7.8).

Now the integral curves in $S(M)$ are the horizontal lifts of the curves in M; that is, these curves are obtained by parallel translating v around the curves in M. The endpoints of the curve through $(m,\, v)$-dotted in Figure 7.9-essentially differ by an element of S^1, namely the rotation $g = e^{i\theta}$, which sends v into its parallel translate around the rectangle in M.

Since the area of the rectangle in M is approximately $\sqrt{s} \cdot \sqrt{s} = s$, the

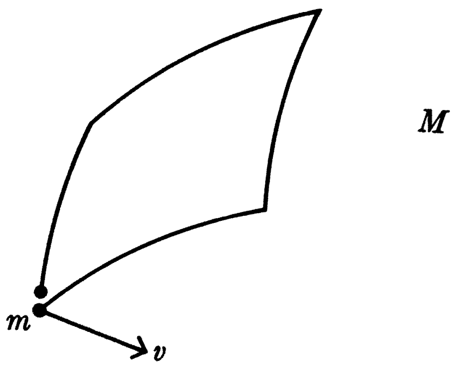

Figure 7.8

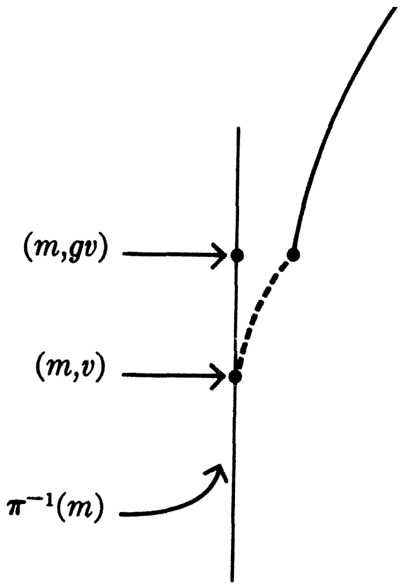

Figure 7.9

limit as $s \to 0$ of the angle of rotation θ divided by the area of the rectangle is equal to the coefficient of V, namely $K(m)$.

Stated precisely and in somewhat greater generality, the result we have been discussing is as follows.

Theorem 1 Let M be an oriented Riemannian 2-manifold. Let $\langle s \rangle$ be an oriented 2-simplex in $\mathbb{R}^2$, and let $h : [s] \to M$ be a map which has a smooth extension mapping a neighborhood of $[s]$ into M. Let α be the closed broken C^∞ curve in M obtained by restricting h to $\partial\langle s \rangle$. Then the rotation obtained by parallel translation around the closed curve α is

$$e^{i \int_{\langle s \rangle} h^*(K\,\mathrm{vol})}$$

so that the angle of rotation is $\displaystyle\int_{\langle s \rangle} h^*\left(K\,\mathrm{vol}\right)$.

Remark Note that this result contains the result discussed above. To obtain $K(m)$, take the limit of $\displaystyle\int_{\langle s \rangle} h^*\left(K\,\mathrm{vol}\right) / \int_{\langle s \rangle} h^*\left(\mathrm{vol}\right)$ as $\langle s \rangle$ shrinks to zero and $h\left(\langle s \rangle\right)$ shrinks to m. However, the theorem says more. For example, it is possible to have $K > 0$ on $h\left([s]\right)$ and still get a trivial rotation upon parallel translating around α, namely when the total angle of rotation $\displaystyle\int_{\langle s \rangle} h^*\left(K\,\mathrm{vol}\right)$ is an integer multiple of 2π.

Proof of Theorem 1 Let $\langle s \rangle = \langle v_0,\, v_1,\, v_2 \rangle$ for some vertices $v_0,\, v_1,\, v_2$ and let $w_0 \in T(M,\, h(v_0))$ be a unit vector. The lines in $[s]$ through v_1 cover $[s]$; their images under h are curves in M which cover $h([s])$. Let $\tilde{h} : [s] \to S(M)$ be obtained by mapping each of these curves into its horizontal lift in $S(M)$ through

$$(h(v_1),\, w_1) \in S(M),$$

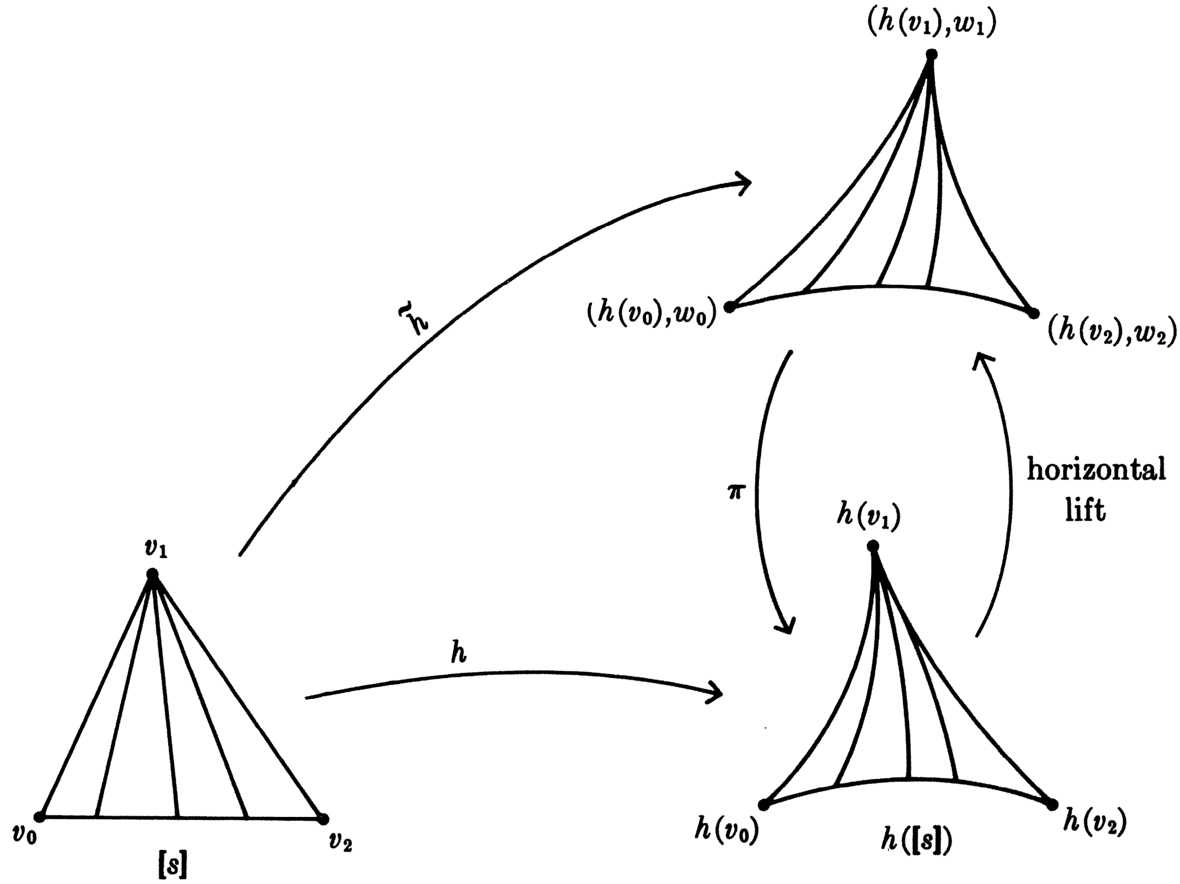

Figure 7.10

where w_1 is the parallel translate of w_0 along the curve $\alpha|_{\langle v_0, v_1\rangle}$ to $h(v_1)$ (Figure 7.10).

By construction, $\pi \circ \tilde{h} = h$. Moreover, $\tilde{h}$ has a smooth extension mapping a neighborhood of $[s]$ into $S(M)$. This may be checked via local coordinates; we omit the computation.

Now

$$\begin{aligned}
\int_{\langle s\rangle} h^*(K\,\mathrm{vol}) &= \int_{\langle s\rangle} \left(\pi \circ \tilde{h}\right)^* (K\,\mathrm{vol}) = \int_{\langle s\rangle} \tilde{h}^* \left[\pi^*(K\,\mathrm{vol})\right] = \int_{\langle s\rangle} \tilde{h}^* \left[(K\circ\pi)\,\omega_1\wedge\omega_2\right] \\
&= -\int_{\langle s\rangle} \tilde{h}^*(d\varphi) \quad \text{(second structural equation)} \\
&= -\int_{\langle s\rangle} d\left(\tilde{h}^*\varphi\right) \\
&= -\int_{\partial\langle s\rangle} \left(\tilde{h}^*\varphi\right) \quad \text{(Stokes's Theorem)} \\
&= -\int_{\partial\langle s\rangle} \varphi\left(d\tilde{h}\left(\frac{d}{dt}\right)\right) dt \\
&= -\int_{\partial\langle s\rangle} \varphi\left(d\tilde{\beta}\left(\frac{d}{dt}\right)\right) dt
\end{aligned}$$

where $\tilde{\beta} = \tilde{h}|_{\partial\langle s\rangle}$. Let $\tilde{\alpha}$ denote the horizontal lift of α through $\tilde{h}(v_0) = (\alpha(v_0), w_0)$. Since $\tilde{\beta}|_{\langle v_0, v_1\rangle} = \tilde{h}|_{\langle v_0, v_1\rangle}$ and $\tilde{\beta}|_{\langle v_1, v_2\rangle} = \tilde{h}|_{\langle v_1, v_2\rangle}$ are horizontal by construction of $\tilde{h}$,

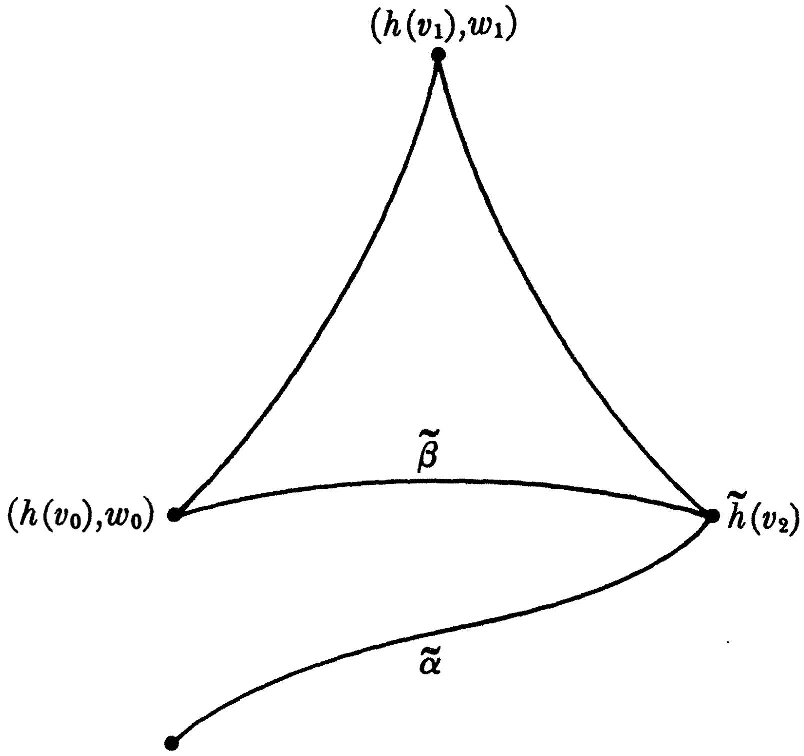

Figure 7.11

we have $\tilde{\beta}|_{\langle v_0, v_1\rangle} = \tilde{\alpha}|_{\langle v_0, v_1\rangle}$ and $\tilde{\beta}|_{\langle v_1, v_2\rangle} = \tilde{\alpha}|_{\langle v_1, v_2\rangle}$. By Lemma 2, Section 7.1, there exists a function $f : \langle v_2, v_0\rangle \to \mathbb{R}$ with $f(v_2) = 0$ such that

$$\tilde{\beta}(t) = e^{if(t)}\,\tilde{\alpha}(t).$$

But $\tilde{\alpha}$ is the horizontal lift of α, so that $\tilde{\alpha}(v_0)$ is the parallel translate of w_0 around α. On the other hand, $\tilde{\beta}(v_0) = w_0$. Hence $e^{if(v_0)}$ is just the rotation mapping the parallel translate of w_0 around α into w_0; that is, $e^{-if(v_0)}$ rotates w_0 into its parallel translate around α. By the second statement of Lemma 2, Section 7.1,

$$\varphi\left(d\tilde{\beta}\left(\frac{d}{dt}\right)\right) = \frac{df}{dt}$$

for $t \in \langle v_2, v_0\rangle$. Moreover, $\varphi\left(d\tilde{\beta}\,(d/dt)\right) = 0$ on $\langle v_0, v_1\rangle$ and $\langle v_1, v_2\rangle$ since β is horizontal there. Thus since $\partial\langle s\rangle = \langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_0\rangle$,

$$\begin{aligned}
\int_{\langle s\rangle} h^*\,(K\,\mathrm{vol}) &= -\int_{\langle v_2, v_0\rangle} \varphi\left(d\tilde{\beta}\left(\frac{d}{dt}\right)\right) dt \\
&= -\int_{\langle v_2, v_0\rangle} \frac{df}{dt}\,dt \\
&= -f(v_0) \\
&= \text{the angle of rotation from } w_0 \text{ to its parallel translate around } \alpha.
\end{aligned}$$

Definitions Let $\alpha : [a, b] \to M$ be a smooth curve. The *length* $\ell(\alpha)$ of α is the real number

$$\ell(\alpha) = \int_a^b \|\dot{\alpha}(t)\|\,dt.$$

The *arc length* along α is the function $s : [a, b] \to \mathbb{R}$ given by

$$s(t) = \int_a^t \|\dot{\alpha}(\tau)\| \, d\tau.$$

Remark l and s are defined because $t \to \|\dot{\alpha}(t)\|$ is continuous. Note that the function s is of class C^1, but it is not necessarily smooth because $t \to \|\dot{\alpha}(t)\|$ is not necessarily differentiable where $\dot{\alpha}(t) = 0$. If, however, $\|\dot{\alpha}(t)\| \neq 0$ for all t, then s is smooth and monotonically increasing.

Definition A curve $\alpha : [a, b] \to M$ is said to be *parameterized by arc length* if $\|\dot{\alpha}(t)\| = 1$ for all $t \in [a, b]$. In this case, $s(t) = t - a$ for all $t \in [a, b]$.

Remark Given any curve $\alpha : [a, b] \to M$ with $\|\dot{\alpha}(t)\| \neq 0$ for all t, a new curve $\alpha_1 : [a, b] \to M$, parameterized by arc length, is obtained by setting

$$\alpha_1 = \alpha \circ s^{-1}.$$

Then $\operatorname{Im} \alpha_1 = \operatorname{Im} \alpha$, and $\ell(\alpha_1) = \ell(\alpha)$.

Remark The concept of arc length extends to broken C^∞ curves a since $\|\dot{\alpha}(t)\|$ is defined at all but a finite number of points.

Definition Given a smooth curve $\alpha : [a, b] \to M$ parameterized by arc length, a smooth curve $\alpha' : [a, b] \to S(M)$ is defined by

$$\alpha'(t) = (\alpha(t), \dot{\alpha}(t)) \quad \text{for } t \in [a, b].$$

α is said to be a *geodesic* in M if α' is horizontal; that is, if α' is the horizontal lift of α through $(\alpha(a), \dot{\alpha}(a)) \in S(M)$. Note that if α is a geodesic, the parallel translate of $\dot{\alpha}(0)$ along α to $\alpha(t)$ is just $\dot{\alpha}(t)$; that is, the tangent to α translates into itself, and α is a "straight line" of the surface.

To measure how far a curve α is from being "straight," we measure how far α' is from being horizontal. Suppose, then, α is parameterized by arc length so that $\alpha' : [a, b] \to S(M)$ is a curve in $S(M)$.

DefinitionThe *geodesic curvature $\kappa_\alpha(t)$* of α at $t \in [a, b]$ is $\psi\left(d\tilde{\alpha}\left(d/dt\right)\right)$ where ψ is the 1-form of the Riemannian connection.

Notation If $\alpha : [a, b] \to M$ is a broken C^∞ curve with $\dot{\alpha}(t) \neq 0$ for all $t \in [a, b]$, let

$$\tau(\alpha) = \int_0^{\ell(\alpha)} \kappa_{\alpha_1}(t) \, dt,$$

where α_1 is the new curve obtained from α by parameterizing by arc length.

If M is a smoothly triangulated manifold, then τ can be considered as a 1-cochain (relative to the triangulation).

Lemma (Gauss-Bonnet Theorem for 2-simplices) Let M be an oriented Riemannian 2-manifold. Let $\langle s\rangle$ be an oriented 2-simplex in $\mathbb{R}^2$, and let $h : [s] \to M$ be a map which has a smooth nonsingular extension mapping a neighborhood of $[s]$ into M. Let α be the closed broken C^∞ curve in M obtained by restricting h to $\partial\langle s\rangle$. Then

$$\int_{\langle s\rangle} h^*\left(K \operatorname{vol}\right) = -\tau\left(\alpha\right) + \sum \left(\text{interior angles of } h([s])\right) - \pi.$$

Proof From Theorem 1 above, $e^{i \int_{\langle s\rangle} h^*(K \operatorname{vol})}$ is the rotation obtained by parallel translation around the closed curve α. Suppose α is broken up into its three smooth curves α_0, α_1 and α_2

so that $\tau(\alpha) = \sum \tau(\alpha_i)$ and $\alpha_i : [a_i, a_{i+1}] \to M$ with $a_0 = a$ and $a_3 = b$. By Lemma 2, Section 7.1, $e^{i\tau(\alpha_i)}$ is the rotation from the parallel translate of $\dot{\alpha}_i(a_i)$ to $\dot{\alpha}_i(a_{i+1})$. Hence, from the picture in M (Figure 7.12), we get that parallel translation around the closed curve α is given by $e^{i(-\tau(\alpha)-\sum \text{exterior angles})}$ Hence, by taking logarithms, we get

$$\int_{\langle s\rangle} h^*(K\,\text{vol}) = -\tau(\alpha) - \sum(\text{exterior angles}) + 2\pi\ell, \quad \text{where } \ell \text{ is an integer.}$$

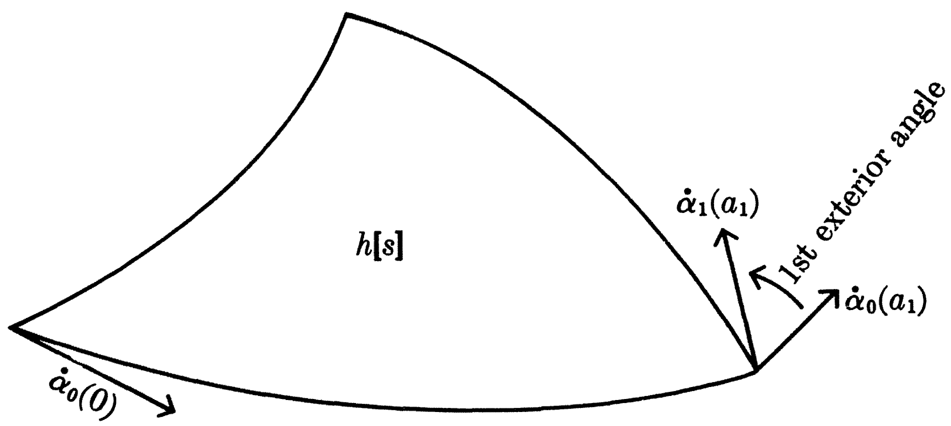

Figure 7.12

We use a continuity argument to show $\ell = 1$. Suppose ρ_0 is a flat Riemannian metric in a neighborhood of $h[s]$ (say transferred from $\mathbb{R}^2$ via h). Then $K = 0$, $\tau(\alpha) = 0$, and $\sum$ (exterior angles) is 2π. Hence for the flat Riemannian metric, $\ell = 1$. Suppose ρ is our given Riemannian metric, and let $\rho_t = t\rho_0 + (1-t)\rho$ be a family of metrics, $t \in [0, 1]$. Let K_t, $\tau_t(\alpha)$, exterior angles$_t$ be the usual entities for ρ_t. These are continuous functions of t. Hence ℓ is a continuous function of t. Since it is an integer for all t and equal to 1 for $t = 0$, we obtain $\ell = 1$. (You can also obtain this result by checking that $\ell = 1$ for small triangles and taking barycentric subdivisions.)

Since interior angle + exterior angle $= \pi$, the lemma is proved.

Definition Let M be an oriented, connected, smoothly triangulated 2-manifold. For each 2-simplex s in M, let $\langle s\rangle$ denote this simplex oriented consistently with M. That is, if $h : K \to M$ is the triangulation and ω is a 2-form on M giving its orientation, let the orientation of s be given by the 2-form $h_s^*\omega$. Let

$$c = \sum_s \langle s\rangle.$$

Then c is a cycle called the *fundamental cycle* of M. Given any 2-form μ on M, the *integral of μ over M* is defined by

$$\int_M \mu = \int_c h_s^*\mu.$$

Exercise Prove that c is a cycle.

Remark The integral can be defined without use of a triangulation. Let M be a compact oriented n-manifold, and let μ be an n-form on M. Let $\{U_j, f_j\}$ be a smooth partition of unity on M, where $\{U_j\}$ is a finite covering of M by coordinate neighborhoods. Then integration of

n-forms is defined on each U_j by pulling the forms back to $\mathbb{R}^n$ through the coordinate systems. The integral of μ over M is then given by

$$\int_M \mu = \sum_j \int_{U_j} f_j \mu.$$

This is independent of the partition of unity used,for if $\{V_k,\, g_k\}$ is another such partition, then

$$\sum_j \int_{U_j} f_j \mu = \sum_{j,\,k} \int_{U_j \cap V_k} f_j g_k \mu = \sum_k \int_{V_k} g_k \mu.$$

Theorem 2 (Gauss-Bonnet Theorem) Let M be an oriented, connected, smoothly triangulated, Riemannian 2-manifold. Then

$$\frac{1}{2\pi} \int_M K \,\text{vol} = \chi(M) = \beta_0 - \beta_1 + \beta_2,$$

where $\chi(M)$ is the Euler characteristic of M.

Proof Note that each 1-simplex t of M is an edge of precisely two 2-simplices of M. For given any point $m \in (t)$, there exists, by the implicit function theorem, a coordinate ball U about m such that $(t) \cap U$ is mapped into a straight line in $\mathbb{R}^2$. By choosing U small enough, t must divide U into precisely two pieces. These pieces must lie in distinct 2-simplices, and, since open simplices are disjoint, there can be no other 2-simplex with t as an edge.

Thus, since each 2-simplex has three 1-simplices as edges, the total number n_1 of 1-simplices of M is given by $n_1 = 3n_2/2$ where n_2 is the number of 2-simplices of M. Letting n_0 denote the number of vertices in M, the Euler characteristic (Section 6.1) is given by

$$\chi = n_0 \,-\, n_1 \,+\, n_2 = n_0 \,-\, (3n_2/2) \,+\, n_2 = n_0 \,-\, (n_2/2)\,.$$

Now we apply the previous lemma, and

$$\begin{aligned} \frac{1}{2\pi} \int_M K \,\text{vol} &= \frac{1}{2\pi} \int_c h^* \left(K \,\text{vol}\right) \\ &= \frac{1}{2\pi} \sum_s \int_{\langle s \rangle} h^* \left(K \,\text{vol}\right) \\ &= \frac{1}{2\pi} \sum_s \Big(-\tau \left(\partial \langle s \rangle\right) + \sum \left(\text{interior angles of } h[s]\right) - \pi\Big) \\ &= \frac{1}{2\pi} \left(-\tau \left(\partial c\right) + \sum_s \Big(\sum \text{interior angles of } h[s]\Big) - n_2 \pi\right). \end{aligned}$$

But $\partial c = 0$, and $\sum_s \Big(\sum \text{interior angles of } h[s]\Big)$ equals the sum over all vertices v in M of the sum of the interior angles at v of all 2-simplices with v as a vertex. Taking a coordinate neighborhood of V contained in $\text{St}\,(v)$, we see that for each v, the sum of these interior angles at v is exactly 2π (Figure 7.13). Hence

$$\frac{1}{2\pi} \int_M K \,\text{vol} = \frac{1}{2\pi} \left(2\pi n_0 \,-\, n_2 \pi\right) = n_0 \,-\, \frac{n_2}{2} = \chi(M).$$

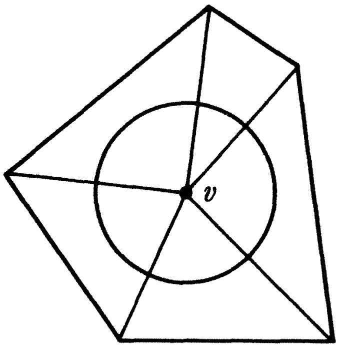

Figure 7.13

Remark Note that this theorem holds for any connection on $S(M)$, since only the second structural equation was used in the proof.

Corollary 1 Let M be any Riemannian 2-manifold homeomorphic with the sphere S^2. Then

$$\int_M K \, \mathrm{vol} = 4\pi.$$

Corollary 2 Let M be any Riemannian manifold homeomorphic with the torus $S^1 \times S^1$. Then

$$\int_M K \, \mathrm{vol} = 0.$$

Corollary 3 Let M be as in the theorem. Suppose there exists on M a smooth vector field which is never zero. Then $\chi(M) = 0$. In particular, there exists no nonzero vector field on S^2.

Proof Let X be such a vector field and let $Y = X/\|X\|$. Then Y is smooth, and $\|Y(m)\| = 1$ for all $m \in M$, so Y is a smooth map $M \to S(M)$. On $S(M)$,

$$d\psi = -(K \circ \pi)\,\omega_1 \wedge \omega_2 = -\pi^*(K \, \mathrm{vol}).$$

Hence

$$\begin{aligned} d(Y^*\psi) &= Y^*(d\psi) = -Y^* \circ \psi^*(K \, \mathrm{vol}) \\ &= -(\pi \circ Y)^*(K \, \mathrm{vol}) \\ &= -K \, \mathrm{vol}, \end{aligned}$$

since $\pi \circ Y = i_M$. Thus $K \, \mathrm{vol}$ is exact and

$$2\pi\chi = \int_c K \, \mathrm{vol} = -\int_c d(Y^*\psi) = -\int_{\partial c} Y^*\psi = 0.$$